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CRT-2002-06

Centre de recherche sur les transports (C.R.T.)
Centre for Research on Transportation

LAGRANGIAN DUALITY AND DECOMPOSITION FOR THE GENERALIZED EQUILIBRIUM PROBLEM

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February 2002

103-03105/

PUBLICATION
CRT-2002-06

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* This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC), and by the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (FCAR).

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Dépôt légal – Bibliothèque nationale du Québec
Bibliothèque nationale du Canada, 2002

Résumé

Dans cet article, nous développons une théorie de la dualité pour le problème d'équilibre général et analysons un algorithme de décomposition pour la résolution de problèmes d'équilibre avec contraintes couplantes.

Keywords: Problème d'équilibre général, dualité, lagrangien, méthodes de décomposition.

Abstract

In this paper, we develop a duality theory for the generalized equilibrium problem, and analyze a decomposition scheme for solving equilibrium problems with coupling constraints.

Keywords: Generalized equilibrium problem, duality theory, Lagrangian, decomposition methods.

1 Introduction

The powerful results of convex duality have sparked off a large body of research on the theoretical and computational aspects pertaining to conjugate duality of variational inequalities (Mosco [18], Fukushima and Itoh [11], Castellani and Mastoeni [7], Robinson [20]), and Lagrangian duality of variational inequalities (Auslender and Teboulle [2], Marcotte and Zhu [17]). In this paper, we present a duality framework for the generalized equilibrium problem (GEP), which subsumes variational inequalities, and propose for the solution of GEPs involving coupling constraints two decomposition schemes whose convergence is proved under suitable regularity conditions.

2 Basic definitions and properties

Let X be a closed convex subset of R^n, Φ a continuous real valued bifunction from $R^n \times R^n$ to R, Θ a mapping from R^n to R^m and C be a nonempty closed convex cone in R^m with vertex at the origin ($\alpha C + \beta C \subseteq C$ for all α, $\beta \geq 0$). The General Equilibrium problem GE(Φ, Γ) (see Fan[15], Mosco [18], Blum and Oettli [6], Bianchi and Schaible [5], Hadjisavvas and Schaible [14]), consists in finding a vector x^* in Γ such that

$$\Phi(x^*, y) \geq 0 \quad \forall y \in \Gamma, \tag{1}$$

where $\Gamma = \{y \in X : \Theta(y) \in -C\}$. Whenever the interior int(C) of C is nonempty, the constraint $\Theta(y) \in -C$ corresponds to inequality constraints. If $C = \{0\}$, all constraints are equality constraints. The variational inequality VI(F, X) is but a special case of GE(Φ, Γ) where $\Phi(x, y) = \langle F(x), y - x\rangle$ and $C = R^n_+$.

Throughout the paper, we will use various concepts of monotonicity for functions, (multi-valued) mappings and bifunctions. We recall that a function F from R^n to R^n is **monotone** on a closed convex subset of R^n if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \forall x, y \in X, \tag{2}$$

strongly monotone on X if there exists a positive number β such that

$$\langle F(x) - F(y), x - y\rangle \geq \beta \|x - y\|^2 \quad \forall x, y \in X, \tag{3}$$

co-coercive on X if there exists a positive number β such that

$$\langle F(x) - F(y), x - y\rangle \geq \beta \|F(x) - F(y)\|^2 \quad \forall x, y \in X, \tag{4}$$

monotone$^+$ on X if F is monotone and

$$\langle F(x) - F(y), x - y\rangle = 0 \Rightarrow F(x) = F(y) \quad \forall x, y \in X. \tag{5}$$

The concepts of co-coercivity and monotonicity$^+$ (see Crouzeix et al. [10]) have been introduced to study the convergence of iterative schemes for solving non strongly monotone variational inequalities (see for instance Zhu and Marcotte [23] and Goffin et al. [13]). These concepts can be extended to multi-valued mappings and bifunctions. A multi-valued mapping H from R^n to R^n is monotone on X if, for all x, y in X, $h_x \in H(x)$, $h_y \in H(y)$:

$$\langle h_x - h_y, x - y\rangle \geq 0 \quad \forall x, y \in X \quad \forall h_x \in H(x), h_y \in H(y). \tag{6}$$

It is **monotone$^+$** on X if it is monotone on X and

$$\langle h_x - h_y, x - y\rangle = 0 \Rightarrow h_y \in H(x) \quad \forall x, y \in X \quad \forall h_x \in H(x), h_y \in H(y). \tag{7}$$

We recall that the mapping H is upper semicontinuous at a point x_0 if

$$\left.\begin{array}{c} x^k \to x^0 \\ h^k \in H(x^k) \\ h^k \to h^0 \end{array}\right\} \Rightarrow h^0 \in H(x^0).$$

For bifunctions, one has the following definitions. A bifunction Φ is **monotone** on X if

$$\Phi(x,y) + \Phi(y,x) \leq 0 \quad \forall x, y \in X. \tag{8}$$

It is **strongly monotone** on X if

$$\Phi(x,y) + \Phi(y,x) \leq -\beta\|x-y\|^2 \quad \forall x, y \in X. \tag{9}$$

The correct generalization of the monotone$^+$ concept is less straightforward, and involves three vectors. We say that the bifunction Φ is **monotone$^+$** on X if it is monotone on X and, for all $x, y \in X$:

$$\Phi(x,y) + \Phi(y,x) = 0 \Rightarrow \Phi(x,y) = \Phi(x,z) - \Phi(y,z) \quad \forall z \in X. \tag{10}$$

The reader interested in various concepts of generalized monotonicity is referred to the papers of Zhu and Marcotte [23], [24] or Tseng [22], and to the references therein.

We say that the mapping Θ is C-**convex** if, for all $x, y \in X$ and $\alpha \in [0,1]$, there holds the inequality

$$\Theta(\alpha x + (1-\alpha)y) - \alpha\Theta(x) - (1-\alpha)\Theta(y) \in -C. \tag{11}$$

Throughout the paper, we assume that the mapping Θ is C-convex on X. This property reduces to the convexity of the individual components of Θ whenever $C = R^n_+$. Under this assumption, the set Γ is convex.

Finally, let $\Pi(x)$ denote the projection of the point x onto the conjugate cone C^* of C: $C^* = \{y : \langle x, y\rangle \leq 0, \forall x \in C\}$. We have:

$$\text{(i)} \quad \langle x - \Pi(x), \Pi(x)\rangle = 0, \tag{12}$$

$$\text{(ii)} \quad \langle x - \Pi(x), y\rangle \leq 0 \qquad \forall y \in C^*, \tag{13}$$

$$\text{(iii)} \quad \langle x - \Pi(x), y - \Pi(x)\rangle \leq 0 \quad \forall y \in C^*, \tag{14}$$

$$\text{(iv)} \quad \|\Pi(x) - \Pi(y)\| \leq \|x - y\|. \tag{15}$$

3 Lagrangian Duality

Let the set X be compact and the function $\Phi(x,\cdot)$ be convex on X, for every $x \in X$. Then the compactness of X, the continuity of Φ and its convexity with respect to the second variable imply, from Ky Fan's theorem [15], that the solution set X^* of GE(Φ, Γ)is nonempty. Moreover, $x^* \in X^*$ if and only if it solves the optimization problem

$$\text{(OP)} \qquad \begin{array}{ll} \min\limits_{y \in X} & \Phi(x^*, y) \\ \text{subject to} & \Theta(y) \in -C. \end{array} \tag{16}$$

The Lagrangian of the optimization problem (OP) is

$$L(y,p) = \Phi(x^*, y) + \langle p, \Theta(y)\rangle \tag{17}$$

where p is the vector of Lagrangian multipliers.

Let us introduce the constraint qualification

$$\text{(CQ)} \qquad \text{int}(C) \neq \emptyset \quad \text{and} \quad \Theta(X) \cap (-\text{int}(C)) \neq \emptyset. \tag{18}$$

If this condition is met, then, according to our assumptions, there exists a saddlepoint of $L(x,p)$ on $X \times C^*$, i.e.,

$$L(x^*,p) \leq L(x^*,p^*) \leq L(x,p^*) \quad \forall x \in X,\ \forall p \in C^*. \tag{19}$$

Lemma 1 *For any $p \in C^*$, the function $\langle p, \Theta(y)\rangle$ is continuous and convex in y on X.*

Proof. The continuity follows from the continuity of Θ. Now, the C-convexity of Θ implies that

$$\Theta(\alpha x + (1-\alpha)y) - \alpha\Theta(x) - (1-\alpha)\Theta(y) \in -C. \tag{20}$$

for every $x, y \in X$ and $\alpha \in [0,1]$. Thus, for any $p \in C^*$, we obtain

$$\langle p, \Theta(\alpha x + (1-\alpha)y)\rangle \leq \langle p, \alpha\Theta(x) + (1-\alpha)\Theta(y)\rangle. \tag{21}$$

and the function $\langle p, \Theta(y)\rangle$ is convex in y on X. The continuity of $\langle p, \Theta(y)\rangle$ follows from the continuity of Θ. □

Based on Lemma 1, the saddlepoint (x^*, p^*) in (19) can be characterized as a solution to the generalized equilibrium systems:

$$\text{(GES)} \qquad \begin{aligned} \Phi(x^*,y) + \langle p^*, \Theta(y) - \Theta(x^*)\rangle &\geq 0, \quad \forall y \in X \qquad (22)\\ \langle \Theta(x^*), p - p^*\rangle &\leq 0, \quad \forall p \in C^*. \qquad (23) \end{aligned}$$

We have just shown that, for any solution x^* of $\text{GE}(\Phi,\Gamma)$, there exists $p^* \in C^*$ such that (x^*, p^*) solves the generalized equilibrium system GES. A natural question is whether any solution of GES is a solution of $\text{GE}(\Phi,\Gamma)$. The next theorem provides a positive answer.

Theorem 1 *Let the constraint qualification (CQ) hold. Then the solution set $X^* \times P^*$ of (GES) is nonempty.*

Proof. Let (x^*, p^*) be a solution of (GES). Since $\langle \Theta(x^*), p - p^*\rangle \leq 0$ for all $p \in C^*$, we have that $\langle \Theta(x^*), p^*\rangle = 0$ and $\Theta(x^*) \in -C$. Hence $x^* \in \Gamma$. Also, from (22), for any $y \in \Gamma$, we have

$$\begin{aligned} \Phi(x^*,y) &\geq \Phi(x^*,y) + \langle p^*, \Theta(x) - \Theta(x^*)\rangle \\ &\geq 0, \end{aligned}$$

which shows that x^* solves $\text{GE}(\Phi,\Gamma)$. □

Before defining the Lagrangian dual mapping, we give some properties of the parametric generalized equilibrium problem GE(p): Find $x(p) \in X$ such that

$$\text{(GE}(p)\text{)} \qquad \Phi(x(p),y) + \langle p, \Theta(y) - \Theta(x(p))\rangle \geq 0 \quad \forall y \in X. \tag{24}$$

Proposition 1 *The function Θ is Lipschitz continuous and the solution set $S(p)$ of the parametric Generalized Equilibrium Problem* GE(p) *is given by*

$$S(p) = \{x(p) \in X : \Phi(x(p),y) + \langle p, \Theta(y) - \Theta(x(p))\rangle \geq 0, \quad \forall y \in X\}.$$

Proof. This is a direct consequence of Lemma 1 and Ky Fan's theorem. □

Based on the above result we define, unambiguously, the **dual mapping**

$$\Omega(p) = \{\Theta(x(p)) : x(p) \in S(p)\} \tag{25}$$

and present some of its continuity and monotonicity properties.

Theorem 2 *If the function Φ is monotone (respectively monotone$^+$) on X, then the mapping $-\Omega$ is monotone (respectively monotone$^+$) on C^*.*

Proof. Let $x(p)$ and $x(q)$ be the respective partial (primal) solutions of the generalized equilibrium problems GE(p) and GE(q). We have

$$\Phi(x(p), x(q)) + \langle p, \Theta(y) - \Theta(x(p))\rangle \geq 0, \tag{26}$$
$$\Phi(x(q), x(p)) + \langle q, \Theta(y) - \Theta(x(q))\rangle \geq 0. \tag{27}$$

The addition of those two inequalities yields

$$\langle -[\Theta(x(p)) - \Theta(x(q))], p - q\rangle \geq -[\Phi(x(p), x(q)) + \Phi(x(q), x(p))]. \tag{28}$$

Since Φ is monotone, the right-hand-side of the inequality is nonnegative, i.e., $-\Omega(p)$ is monotone. Assume furthermore that $x(p)$ and $x(q)$ satisfy

$$\langle (\Theta(x(p)) - \Theta(x(q)), p - q\rangle = 0. \tag{29}$$

If Φ is monotone$^+$, this implies that

$$\Phi(x(q), y) = \Phi(x(q), x(p)) + \Phi(x(p), y) \quad \forall y \in X.$$

From the definition of $x(p)$ and $x(q)$ and (29) we infer that

$$\begin{aligned} \Phi(x(q), y) + \langle p, \Theta(y) - \Theta(x(q))\rangle &= \Phi(x(p), y) + \langle p, \Theta(y) - \Theta(x(p))\rangle \\ &\quad + \Phi(x(q), x(p)) + \langle p, \Theta(x(p)) - \Theta(x(q))\rangle \end{aligned}$$

which is nonnegative. Hence $x(q) \in S(p)$ and $\Theta(x(q)) \in \Omega(p)$, i.e., $-\Omega$ is monotone$^+$ on C^*. □

Theorem 3 *If the mapping Φ is strongly monotone with modulus β and Θ is Lipschitz continuous with constant τ, then the dual mapping $-\Omega$ is single-valued and co-coercive with modulus $\alpha = \beta/\tau^2$ on C^*.*

Proof. Adding the inequalities (26) and (27) yields

$$\begin{aligned} \langle p - q, \Theta(x(p)) - \Theta(x(q))\rangle &\geq -\Big[\Phi(x(p), x(q)) + \Phi(x(q), x(p))\Big] \\ &\geq \beta\|x(p) - x(q)\|^2 \\ &\geq \frac{\beta}{\tau^2}\|\Theta(x(p)) - \Theta(x(q))\|^2 \end{aligned}$$

which shows that $-\Omega$ is co-coercive with modulus $\alpha = \beta/\tau^2$. □

Theorem 4 *If Θ is Lipschitz continuous on X, the mappings S and Ω are upper-semicontinuous on C^*.*

Proof. Let $\{p^k\}$ be a sequence such that $p^k \in C^*$ and $p^k \to p^0 \in C^*$. Take $r^k \in S(p^k)$ such that $r^k \to r^0$. Since $r^k \in S(p^k)$, we have

$$\Phi(r^k, y) + \langle p^k, \Theta(y) - \Theta(r^k)\rangle \geq 0 \quad \forall y \in X. \tag{30}$$

Taking the limit in (30), the continuity of ϕ and Θ imply that

$$\Phi(r^0, y) + \langle p^0, \Theta(y) - \Theta(r^0)\rangle \geq 0 \quad \forall y \in X. \tag{31}$$

Since X is closed, $r^0 \in X$ and $r^0 \in S(p^0)$. Therefore S is upper-semicontinuous.

Now, assume that $p^k \in C^*$, $p^k \to p^0$, $x(p^k) \in S(p^k)$, $s^k = -\Theta(x(p^k))$ and $s^k \to s^0$. Since X is compact, then there exists a subsequence $\{x(p^{k'})\} \subset \{x(p^k)\}$ such that $x(p^{k'}) \to r^0 \in X$ and $s^0 = -\Theta(r^0)$. However, since S is upper-semicontinuous, $r^0 \in S(p^0)$ and $s^0 \in \Omega(p^0)$. We conclude that the mapping Ω is also upper-semicontinuous. □

4 Dual-based Decomposition Methods

In this section, we consider the problem $\mathrm{GE}(\Phi, \Gamma)$ where the the set X takes the separable form

$$X = X_1 \times ... \times X_I, \tag{32}$$

and where the 'complicating' or coupling constraints are defined by the function Θ. We first address the case where the bifunction Φ and the function Θ are separable, i.e.,

$$\begin{aligned} \Phi(x, y) &= \sum_{i=1}^{I} \Phi(x_i, y_i) \\ \Theta(x) &= \sum_{i=1}^{I} \Theta_i(x_i). \end{aligned}$$

To solve this problem, we consider a primal-dual iterative scheme.

PRIMAL-DUAL ALGORITHM 1 (separable case)

PRIMAL STEP

Find x^k such that

$$\Phi(x^k, x) + \langle p^k, \Theta(x) - \Theta(x^k)\rangle \geq 0, \quad \forall x \in X \tag{33}$$

DUAL STEP

$$p^{k+1} = \Pi(p^k + \rho\Theta(x^k)) \tag{34}$$

Theorem 5 *Assume that*

(i) $\Phi(x, y)$ *is strongly monotone with modulus* β *on* X,

(ii) Θ *is Lipschitz continuous with constant* τ *on* X,

(iii) *the constraint qualification (CQ) is satisfied,*

(iv) $0 < \rho < 2\beta/\tau^2$.

Then any sequence $\{(x^k, p^k)\}$ *generated by the primal-dual algorithm is bounded, the entire sequence* $\{x^k\}$ *converges to some equilibrium point* x^* *and, for any cluster point* $\bar{p}$ *of the sequence* $\{p^k\}$, $(x^*, \bar{p})$ *is a solution of the primal-dual system (GES) over* $X \times C^*$.

Proof. From the assumptions and Theorem 3, the solution x^k obtained at the primal step of the algorithm is unique. Letting $x^* \in X^*$ in (33) and $x = x^k$ in (22) yields

$$\begin{aligned} \langle p^k - p^*, \Theta(x^*) - \Theta(x^k)\rangle &\geq -(\Phi(x^k, x^*) + \Phi(x^*, x^k)) \\ &\geq \beta\|x^k - x^*\|^2. \end{aligned} \tag{35}$$

From the property of the projection and (39), we derive

$$\|p^{k+1} - p^*\|^2 \leq \|p^k - p^* + \rho(\Theta(x^k) - \Theta(x^*))\|.$$

From the assumptions and (35), it follows that

$$\begin{aligned} \|p^{k+1} - p^*\|^2 &\leq \|p^k - p^*\|^2 + 2\langle p^k - p^*, \Theta(x^k) - \Theta(x^*)\rangle + \rho^2\|\Theta(x^k) - \Theta(x^*)\|^2 \\ &\leq \|p^k - p^*\|^2 + \rho(\rho\tau^2 - 2\beta)\|x^k - x^*\|^2. \end{aligned}$$

If $p^{k+1} = p^k$ for some k, then , from (33) and (34), it is clear that $(x^k, p^k) \in X^* \times P^*$ and the algorithm stops finitely with a primal-dual solution. Otherwise, the sequence of positive numbers $\{\|p^k - p^*\|\}$ is strictly decreasing, the sequence $\{x^k\}$ has a limit point x^*. Furthermore, the sequence $\{p^k\}$ is bounded and its cluster points belong to C^*. Let $\bar{p}$ be an arbitrary cluster point of the sequence $\{p^k\}$ and $\{p^{k_i}\}_{i\in L}$ a subsequence converging to a $\bar{p}$. By taking the limit over the indices $i \in L$ in (33) and (34) we obtain

$$\Phi(x^*, x) + \langle \bar{p}, \Theta(x) - \Theta(x^*)\rangle \geq 0, \quad \forall x \in X, \tag{36}$$

$$\langle \Theta(x^*), p - \bar{p}\rangle \leq 0, \quad \forall p \in C^*. \tag{37}$$

Therefore, $(x^*, \bar{p}) \in X^* \times P^*$ is a solution of the primal-dual system (GES). □

The algorithm for the non separable case is based on an approximation of the bifunction Φ. It implements the auxiliary problem principle of Cohen [8], Cohen and Zhu [9] within a primal-dual setting which replaces, at each iteration, a general equilibrium problem by a strongly convex mathematical program.

PRIMAL-DUAL ALGORITHM 2 (non separable case)

PRIMAL STEP

Find x^{k+1} such that

$$\langle \epsilon \nabla \Phi_y(x^k, x^k) + K'(x^{k+1}) - K'(x^k), x - x^{k+1} \rangle + \epsilon \langle p^k, \Theta(x) - \Theta(x^{k+1}) \rangle \geq 0 \quad \forall x \in X \tag{38}$$

DUAL STEP

$$p^{k+1} = \Pi(p^k + \rho \Theta(x^k)) \tag{39}$$

In the algorithm, ϵ is any positive number and K denotes a strongly convex differentiable function with gradient K'. From the point of view of decomposition, the interest of this algorithm is twofold. First, GE(Φ, Γ) is replaced by a strongly convex mathematical program, for which there exists a wealth of efficient methods. Next, if the ground set X is separable and the function Θ is additive, then the primal step will involve a separable optimization problem, whenever the approximation function is itself separable. More precisely, if $\Theta(x) = \sum_{i=1}^{I} \Theta_i(x_i)$ and $K(x) = \sum_{i=1}^{I} K_i(x_i)$, subproblem (38) splits into I independent subproblems:

$$\langle \epsilon \nabla \Phi_{yi}(x^k, x_i^k) + K_i'(x_i^{k+1}) - K_i'(x_i^k), x_i - x_i^{k+1} \rangle + \epsilon \langle p^k, \Theta_i(x_i) - \Theta_i(x_i^{k+1}) \rangle \geq 0, \quad \forall x_i \in X_i.$$

Theorem 6 *Assume that*

(i) $\Phi(x, y)$ *is continuously differentiable and strongly monotone with modulus* β *on* X.

(ii) *The partial gradient of* ϕ *with respect to* y *are Lipschitz continuous, i.e.,*

$$\|\nabla_y \Phi(x, y_1) - \nabla_y \phi(x, y_2)\| \leq B \|y_1 - y_2\| \tag{40}$$

$$\|\nabla_y \Phi(x_1, y) - \nabla_y \phi(x_2, y)\| \leq A \|x_1 - x_2\|, \tag{41}$$

(iii) Θ *is Lipschitz continuous with constant* τ *on* X,

(iv) *the constraint qualification (CQ) is satisfied,*

(v) K *is strongly convex with constant* b *and differentiable with a Lipschitz continuous gradient.*

Then the solution of (38) exists and is unique. Moreover if

(vi) $0 < \epsilon < \frac{b}{(A^2/\beta) + B + \rho\tau^2}$ *and*

(vii) $0 < \rho < \frac{\beta}{\tau^2}$

the sequence (x^k, p^k) *generated by Algorithm 2 is bounded, the entire sequence* $\{x^k\}$ *converges to some equilibrium point* x^* *and, for any cluster point* $\bar{p}$ *of the sequence* $\{p^k\}$, $(x^*, \bar{p})$ *is a solution of the primal-dual system (GES) over* $X \times C^*$.

Proof. Since K is strongly convex, there exists a unique solution of (38). We can write:

$$\begin{aligned}
& \langle K'(x^{k+1}) - K'(x^k), x^* - x^{k+1}\rangle + \epsilon(\Phi(x^k, x^*) - \Phi(x^k, x^k)) \\
& + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^k - x^{k+1}\rangle + \epsilon\langle p^k, \Theta(x^*) - \Theta(x^{k+1})\rangle \\
\geq\ & \langle K'(x^{k+1}) - K'(x^k), x^* - x^{k+1}\rangle + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^* - x^k\rangle \\
& + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^k - x^{k+1}\rangle + \epsilon\langle p^k, \Theta(x^*) - \Theta(x^{k+1})\rangle \\
\geq\ & \langle K'(x^{k+1}) - K'(x^k), x^* - x^{k+1}\rangle \\
& + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^* - x^{k+1}\rangle + \epsilon\langle p^k, \Theta(x^*) - \Theta(x^{k+1})\rangle \\
\geq\ & 0 \quad \forall x \in X, \text{ since } x^{k+1} \text{ solves (38).}
\end{aligned}$$

Let us add ϵ times (22), with $y = x^{k+1}$, to the previous inequality, to obtain

$$\begin{aligned}
& \langle K'(x^{k+1}) - K'(x^k), x^* - x^{k+1}\rangle + \epsilon(\Phi(x^k, x^*) + \Phi(x^*, x^{k+1})) \\
& + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^k - x^{k+1}\rangle + \epsilon\langle p^k - p^*, \Theta(x^*) - \Theta(x^{k+1})\rangle \ \geq\ 0 \quad \forall x \in X. \qquad (42)
\end{aligned}$$

Now, the strong convexity of K allows us to write the first term of (42) as

$$\begin{aligned}
\langle K'(x^k) - K'(x^{k+1}), x^* - x^{k+1}\rangle &= -\langle K'(x^{k+1}), x^* - x^{k+1}\rangle \\
&\quad + \langle K'(x^k), x^* - x^k\rangle + \langle K'(x^k), x^k - x^{k+1}\rangle \\
&\geq [K(x^*) - K(x^{k+1}) - \langle K'(x^{k+1}), x^* - x^{k+1}\rangle] \\
&\quad - [K(x^*) - K(x^k) - \langle K'(x^k), x^* - x^k\rangle] + \frac{b}{2}\|x^k - x^{k+1}\|^2
\end{aligned}$$

and the sum of the next two terms as

$$\begin{aligned}
& \epsilon(\Phi(x^k, x^*) + \Phi(x^*, x^{k+1})) + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^k - x^{k+1}\rangle \\
& \leq \epsilon[(\Phi(x^k, x^*) + \Phi(x^*, x^k)) + (\Phi(x^*, x^{k+1}) - \Phi(x^*, x^k))] + \epsilon\langle \nabla\Phi_y(x^k, x^k), x^k - x^{k+1}\rangle \\
& \leq \epsilon[-\beta\|x^k - x^*\|^2 + \langle \nabla\Phi_y(x^*, x^k) - \nabla\Phi_y(x^k, x^k), x^{k+1} - x^k\rangle| + \frac{B}{2}\|x^k - x^{k+1}\|^2] \\
& \leq \epsilon[-\beta\|x^k - x^*\|^2 + A\|x^k - x^*\|\|x^{k+1} - x^k\| + \frac{B}{2}\|x^{k+1} - x^k\|^2] \\
& \leq \epsilon[-\frac{\beta}{2}\|x^k - x^*\|^2 + \frac{A^2/\beta + B}{2}\|x^{k+1} - x^k\|^2].
\end{aligned}$$

Now, it follows from Property (15) of the projection that

$$\|p^{k+1} - p^*\|^2 \ \leq\ \|p^k - p^* + \rho(\Theta(x^{k+1}) - \Theta(x^*))\|^2.$$

Expanding the square on the right-hand-side, and using the Lipschitz property of the mapping Θ we obtain:

$$\begin{aligned}
2\langle p^k - p^*, \Theta(x^*) - \Theta(x^{k+1})\rangle \ &\leq\ \frac{1}{\rho}(\|p^k - p^*\|^2 - \|p^{k+1} - p^*\|^2) \\
&+\ \rho\tau^2\|x^{k+1} - x^*\|^2, \qquad (43)
\end{aligned}$$

which provides an upper bound for the fourth term of (42). Regrouping all these inequalities leads to

$$\begin{aligned}
&[K(x^{k+1}) + \langle K'(x^{k+1}), x^* - x^{k+1}\rangle] - [K(x^k) + \langle K'(x^k), x^* - x^k\rangle] \\
&+ \frac{\epsilon((A^2/\beta) + B + \rho\tau^2) - b}{2}\|x^k - x^{k+1}\|^2 + \frac{\epsilon(\rho\tau^2 - \beta)}{2}\|x^k - x^*\|^2 \\
&- \frac{\epsilon}{2\rho}(\|p^k - p^*\|^2 - \|p^{k+1} - p^*\|^2) \geq 0. \qquad (44)
\end{aligned}$$

Now we introduce the 'distance function'

$$G(x,p) = K(x^*) - K(x) - \langle K'(x), x^* - x\rangle + \frac{\epsilon}{2\rho}\|p - p^*\|^2 \qquad (45)$$

which, in view of (44), satisfies

$$\begin{aligned}
G(x^k,p^k) - G(x^{k+1},p^{k+1}) \;\geq\; & \frac{b - \epsilon((A^2/\beta) + B + \rho\tau^2)}{2}\|x^k - x^{k+1}\|^2 \\
& + \frac{\epsilon(\beta - \rho\tau^2)}{2}\|x^k - x^*\|^2. \qquad (46)
\end{aligned}$$

The assumptions of the theorem imply that the sequence $\{G(x^k,p^k)\}$ is decreasing. Moreover, it is bounded from below since

$$G(x,p) \geq \frac{\epsilon}{\rho}\|p - p^*\|^2 + \frac{b}{2}\|x - x^*\|^2 \geq 0.$$

Therefore, the sequence $\{G(x^k,p^k)\}$ converges, and the left-hand side of (46) goes to zero when $k \to \infty$. This implies, since the right-hand side of (46) is nonnegative, that $\|x^k - x^*\| \to 0$.

Note that, since $G(x^k,p^k)$ and x^k possesses limits, so does $\|p^k - p^*\|$, and the sequence $\{p^k\}$ is bounded. Let $\bar{p}$ be a cluster point of the sequence $\{p^k\}$ and $\{p^{k'}\}_{k' \in L}$ a subsequence converging to $\bar{p}$. Since C^* is closed and convex, $\bar{p} \in C^*$ and, from the previous results,

$$\langle p^{k'}, \Theta(x^{k'})\rangle \to \langle \bar{p}, \Theta(x^*)\rangle \leq 0.$$

On the other hand, the projection p^{k+1} satisfies the variational inequality

$$\langle p - p^{k+1}, p^k + \rho\Theta(x^{k+1}) - p^{k+1}\rangle \leq 0 \quad \forall p \in C^*.$$

Setting $p = p^*$ in the above, we obtain

$$\begin{aligned}
\rho\langle p^{k+1} - p^*, \Theta(x^{k+1})\rangle \;&\geq\; \langle p^{k+1} - p^*, (p^{k+1} - p^*) - (p^k - p^*)\rangle \\
&\geq\; \frac{1}{2}(\|p^{k+1} - p^*\|^2 - \|p^k - p^*\|^2).
\end{aligned}$$

Considering the inequality for indices $k' - 1$, and recalling that $\|p^k - p^*\|^2$ has a limit, we obtain

$$\langle \bar{p}, \Theta(x^*)\rangle - \langle p^*, \Theta(x^*)\rangle \geq 0.$$

Thus,

$$\langle \bar{p}, \Theta(x^*)\rangle = 0 \qquad \text{and } \langle \Theta(x^*), p - \bar{p}\rangle \leq 0, \ \forall p \in C^*.$$

By taking the limit over the indices $k \in L$ in (38) we obtain

$$\langle \nabla \Phi_y(x^*, x^*), x - x^* \rangle + \langle \bar{p}, \Theta(x) - \Theta(x^*) \rangle \geq 0 \quad \forall x \in X.$$

From the convexity of $\Phi(x^*, .)$ we get

$$\Phi(x^*, x) + \langle \bar{p}, \Theta(x) - \Theta(x^*) \rangle \geq 0, \quad \forall x \in X.$$

Therefore, $(x^*, \bar{p}) \in X^* \times P^*$ is a solution of the primal-dual system (GES). □

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